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REGENERATIVE SIMULATION OF RESPONSE TIMES IN NETWORKS OF QUEUES, II: MULTIPLE JOB TYPES

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and

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REGENERATIVE SIMULATION OF RESPONSE TIMES IN NETWORKS OF QUEUES, II: MULTIPLE JOB TYPES

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ABSTRACT: We have previously discussed the simulation of networks of queues for general characteristics of passage times of a single job type, using the regenerative method for simulation and the idea of tracking a distinguished job through the network. We consider here,
In this paper they consider
from a somewhat different point of view, passage time simulation in closed networks of queues having multiple job types. ~~Our~~ results
Their
provide a means of obtaining, from a single replication, point and interval estimates for passage times of the several job types. They also yield a statistically more efficient estimation procedure for passage times of a single job type.

B

1. INTRODUCTION

We have considered in two previous papers (Iglehart and Shedler, [9], [10]) the problem of simulating closed and finite capacity open networks of queues, respectively, for general characteristics of passage times. A passage time, defined formally in Section 3, is the time for a job to traverse a portion of the network. In general, the calculation of passage times comprises random sums of queueing times. In a closed network of queues, when the passage time is a complete circuit or loop, we refer to it as a response time. This paper presents a method for estimation of passage times in networks having multiple job types, i.e., networks with stochastically non-identical jobs.

For closed networks, we introduced in [9] the notion of a distinguished "marked" job. After arbitrarily selecting a job to serve as the marked job, we observe in the simulation the times at which passage (or response) times of this job start and terminate. These observations are the basis for construction of confidence intervals for the quantities of interest associated with the limiting passage time. In [9] we considered passage times in networks with a single job type. Under consideration here are networks with multiple job types and the estimation of individual and joint characteristics of passage times over the several job types. The type of a job may influence its routing through the network as well as its service requirements at each center. For expository convenience, we assume that there are only two job types in the network and we mark one job of each type. By tracking these two jobs, we are able to produce from a single replication confidence intervals for a variety of passage time

characteristics. The method of this paper can also be applied to networks with only a single job type; the result is an alternative estimation scheme to that proposed in [9].

2. NETWORKS OF QUEUES AND ASSOCIATED STOCHASTIC PROCESSES

We consider closed networks of queues with a finite number of jobs, N, of two types, and assume that there are N_1 [resp. N_2] jobs of type 1 [resp. type 2] with $N_1+N_2=N$. In each network there are a finite number of service centers, s, and a finite number of job classes, c. All jobs retain their job type, but may change class as they traverse the network. (Think of type 1 jobs as cubes and type 2 jobs as spheres, and let job classes correspond to different colors. Then we permit jobs to change color, but not shape.) Upon completion of service at center i, a type ν job of class j goes to center k and changes to class ℓ with probability $p_{ij,k\ell}^{(\nu)}$. We assume that for $\nu=1,2$, $\underline{P}^{(\nu)}=\{p_{ij,k\ell}^{(\nu)}\colon 1\le i,k\le s, 1\le j,\ell\le c\}$ is a given irreducible Markov matrix.

The service times and service discipline at each service center are as in [9] with the exception that they may also depend on job type. We briefly review the situation. At each service center jobs queue and receive service according to a fixed priority scheme among classes and types, which scheme can vary from center to center. Each center operates as a single server, processing jobs of a fixed type and class according to a fixed service discipline. All service times in the network are mutually independent, and at each center have a distribution with an exponential stage representation (Cox [2], p. 314) with parameters which

may depend on the service center, type and class of job being serviced, and the "state" of the entire system. (We exclude zero service times occurring with positive probability.) A job in service may or may not be pre-empted (according to a fixed procedure for each center) if another job of higher priority joins the queue at the center.

We restrict the present discussion to networks in which all service times are exponentially distributed, and deal with distributions having an exponential stage representation in the usual way by the method of stages (cf., e.g., Gelenbe and Muntz [3]). To characterize the state of the system at time t, we let $S_i(t)$ denote the (type, class) pair of the job receiving service at center i at time t, $i=1,2,\ldots,s$. If there are no jobs at center i at time t, we set $S_i(t)=(0,0)$. We denote by $j_1(i),\ldots,j_{k(i)}(i)$ the (type, class) pairs served at center i ordered by decreasing priority, and let $C_{j_1}^{(i)}(t),\ldots,C_{j_{k(i)}}^{(i)}(t)$ denote the number of jobs in queue at time t of the various (type, class) pairs served at center i. We mark one job of each of the two types in order to measure their passage or response times. As in [9], we view the N jobs as being completely ordered in a linear stack, and let the vector Z(t) be given by:

$$Z(t) = (C_{j_{k(1)}}^{(1)}(t),\ldots,C_{j_1}^{(1)}(t),S_1(t);\ldots;$$
$$C_{j_{k(s)}}^{(s)}(t),\ldots,C_{j_1}^{(s)}(t),S_s(t)) \quad .$$

The linear stack again corresponds to the order of components in the vector Z(t) after ignoring any zero components. Within a (type, class) pair at a center, jobs waiting appear in the linear stack in the order of their

arrival in the center, the latest to arrive being closest to the top of the stack. Let $N_\nu(t)$, $\nu=1,2$, denote the position from the top in this linear stack of the type ν marked job. Then the state vector of the network is

$$X(t) = (Z(t), N_1(t), N_2(t)), \quad t\geq 0 \quad .$$

Under the exponential service time and Markovian routing assumptions, the process $\underset{\sim}{X}=\{X(t):t\geq 0\}$ is an irreducible continuous time Markov chain with finite state space E.

3. SIMULATION FOR PASSAGE TIMES

We specify the passage (or response) times for the two types of jobs by eight subsets of E: $A_1^{(\nu)}$, $A_2^{(\nu)}$, $B_1^{(\nu)}$, $B_2^{(\nu)}$, for $\nu=1,2$. The sets $A_1^{(\nu)}$ and $A_2^{(\nu)}$ [resp. $B_1^{(\nu)}$, $B_2^{(\nu)}$] determine when to start [resp. stop] the clock measuring a particular passage time for the type ν marked job. We denote the jump times of $\underset{\sim}{X}$ by $\{\eta_n:n\geq 0\}$, and in terms of these, we define four sequences of random times: $\{S_j^{(\nu)}:j\geq 0\}$ and $\{T_j^{(\nu)}:j\geq 1\}$, for $\nu=1,2$. The start [resp. termination] time of the jth passage time for the type ν marked job is denoted by $S_{j-1}^{(\nu)}$ [resp. $T_j^{(\nu)}$]. Formally, we have for $\nu=1,2$,

$$S_j^{(\nu)} = \inf\{\eta_n \geq T_j^{(\nu)}: \ X(\eta_n)\in A_2^{(\nu)}, \ X(\eta_{n-1})\in A_1^{(\nu)}\}, \ j\geq 0$$

$$T_j^{(\nu)} = \inf\{\eta_n > S_{j-1}^{(\nu)}: \ X(\eta_n)\in B_2^{(\nu)}, \ X(\eta_{n-1})\in B_1^{(\nu)}\}, \ j\geq 1 \quad .$$

The jth passage time for the type ν marked job is $P_j^{(\nu)}=T_j^{(\nu)}-S_{j-1}^{(\nu)}$, $j\geq 1$. Note that the definition of these times is a special case of the corresponding times defined in [9]. This specialization still allows us

to deal with passage times in most networks of interest. For response times of type ν jobs, $A_1^{(\nu)}=B_1^{(\nu)}$, $A_2^{(\nu)}=B_2^{(\nu)}$, and $S_j^{(\nu)}=T_j^{(\nu)}$ for all $j\geq 1$.

At this point we depart from the method given in [9]. Let L(t) denote the last state visited by the Markov chain $\underset{\sim}{X}$ before jumping to X(t), and set

$$V(t) = (L(t),X(t)),\ t\geq 0\ \ .$$

The process $\underset{\sim}{V}=\{V(t):t\geq 0\}$ has a state space F consisting of all pairs of states (i,j), $i,j\in E$ for which a transition in $\underset{\sim}{X}$ from state i to state j can occur with positive probability. In general, of course, the size of the state space F is larger than that of E. The "Q-matrix" used in generating the Markov chain $\underset{\sim}{V}$ can be obtained easily from that for $\underset{\sim}{X}$. Since $\underset{\sim}{X}$ is an irreducible, positive recurrent Markov chain, so is $\underset{\sim}{V}$. Clearly, the entrance times of $\underset{\sim}{V}$ to a state $(i,j)\in F$ correspond to the times of transition in $\underset{\sim}{X}$ from state i to state j. For a type ν job, we define two subsets of F according to:

$$S^{(\nu)} = \{(i,j)\in F:\ \ i\in A_1^{(\nu)},\ j\in A_2^{(\nu)}\}$$

$$T^{(\nu)} = \{(i,j)\in F:\ \ i\in B_1^{(\nu)},\ j\in B_2^{(\nu)}\}\ \ .$$

Thus the entrances of $\underset{\sim}{V}$ to $S^{(\nu)}$ [resp. $T^{(\nu)}$] correspond to the start [resp. termination] times of passage times for the type ν marked job. Of course for response times of a type ν job, $S^{(\nu)}=T^{(\nu)}$. This new set-up involving the process $\underset{\sim}{V}$ permits a more straightforward consideration of passage (or response times) than was the case in [9].

The argument employed in [8], Appendix 1, shows that for $\nu=1,2$, $P_n^{(\nu)}$ converges in distribution to a random variable $P^{(\nu)}$, denoted $P_n^{(\nu)} \Longrightarrow P^{(\nu)}$ as $n\to\infty$, and that the sequence of passage times for any other job of type ν also converges in distribution to the same random variable $P^{(\nu)}$. Moreover, the sequence of passage times of type ν jobs (irrespective of job identity) in the order of start (or termination) also converges in distribution to $P^{(\nu)}$. Our concern is with the estimation of characteristics associated with these limiting passage times.

Estimation of $E\{R^{(1)}\}$ and $P\{R^{(1)}\le x\}$

To illustrate how simple the passage time estimation problem becomes in the setting of the process $\underset{\sim}{V}$, we consider first the estimation of characteristics of a limiting response time, $R^{(1)}$, of a type 1 job. For this estimation problem, of course, it is not necessary to mark a type 2 job. Since $R^{(1)}$ is a response time, $S^{(1)}=T^{(1)}$. We select a fixed state of $S^{(1)}$, which for convenience we designate state 0, and assume that $\underset{\sim}{V}(0)=0$.

Suppose first that we wish to estimate $E\{R^{(1)}\}$. The successive entrances of $\underset{\sim}{V}$ to $S^{(1)}$ constitute the start and termination of response times of the type 1 marked job. Let $R_n^{(1)}$, $n\ge0$, denote the time between the nth and (n+1)st entrances to $S^{(1)}$, with the 0th entrance to $S^{(1)}$ occurring at $t=0$. Also, let $\{V_n:n\ge0\}$ denote the embedded jump chain associated with $\underset{\sim}{V}$. The random times $\{\tau_n:n\ge1\}$ and $\{\delta_n:n\ge1\}$ will denote the lengths of the successive 0-cycles (successive returns to the fixed

state 0) for $\underset{\sim}{V}$ and $\{V_n : n \geq 0\}$, respectively. Then the number of response times for the type 1 marked job in the first 0-cycle of $\underset{\sim}{V}$ is

$$N_1^{(1)} = \sum_{j=0}^{\delta_1 - 1} 1_{\{V_n \in S^{(1)}\}}$$

and the sum of the response times in that cycle is simply

$$\tau_1 = \sum_{n=0}^{N_1^{(1)}-1} R_n^{(1)} \quad .$$

We denote the analogous quantities in the kth 0-cycle by $N_k^{(1)}$ and τ_k. Since $\underset{\sim}{V}$ is a regenerative process, the pairs of random variables $\{(\tau_k, N_k^{(1)}) : k \geq 1\}$ are independent and identically distributed (i.i.d.). Provided that $E\{R^{(1)}\} < \infty$, a renewal argument (cf., [10]) shows that

$$E\{R^{(1)}\} = E\{\tau_1\}/E\{N_1^{(1)}\} \quad . \tag{3.1}$$

At this point, the standard regenerative method arguments (cf., [7]) can be used to construct (based on n cycles) the point estimate $\overline{\tau}/\overline{N}_1^{(1)}$ and an associated confidence interval for $E\{R^{(1)}\}$. Here $\overline{\tau} = (\tau_1 + \ldots + \tau_n)/n$ and $\overline{N}^{(1)} = (N_1^{(1)} + \ldots + N_n^{(1)})/n$. For this problem, we could also employ the discrete time estimation method suggested by Hordijk, Iglehart and Schassberger [6]; this would shorten the computation time and increase statistical efficiency.

If we are interested in the distribution function, $P\{R^{(1)} \leq x\}$ of $R^{(1)}$, we proceed as above, but define in addition the i.i.d. sequence of random variables $\{Y_k : k \geq 1\}$, where, for example,

$$Y_1 = \sum_{n=1}^{N_1^{(1)}} 1_{\{R_n^{(1)} \le x\}} \quad .$$

Then the point estimate of $P\{R^{(1)} \le x\}$ is just $\overline{Y}/\overline{N}^{(1)}$, and we obtain confidence intervals in the usual way.

Estimation of $E\{R^{(1)}\}$ and $E\{R^{(2)}\}$

Now suppose that we wish to estimate the expected passage time for type 2 jobs, $E\{R^{(2)}\}$, as well as $E\{R^{(1)}\}$. Response times for the type 2 marked job start and terminate at the entrance times of $\underset{\sim}{V}$ to the set $S^{(2)}=T^{(2)}$. Let $N_k^{(2)}$ denote the number of entrances to $S^{(2)}$ of $\underset{\sim}{V}$ in the kth 0-cycle. For example, in the first 0-cycle

$$N_1^{(2)} = \sum_{n=0}^{\delta_1 - 1} 1_{\{V_n \in S^{(2)}\}} \quad .$$

Although we are able to begin the simulation at the start of a response time for the type 1 marked job, in general a response time for the type 2 marked job is underway at time t=0. Similarly, at the end of a 0-cycle, a response time for the type 1 marked job terminates, but a response time for the type 2 marked job is still underway. After n 0-cycles, $N_1^{(2)}+\ldots+N_n^{(2)}$ response times for the type 2 marked job have started and the sum of these response times is approximately $\tau_1+\ldots+\tau_n$. The error in this approximation is due to the partial response time at t=0 which is not counted in $N_1^{(2)}+\ldots+N_n^{(2)}$ and the last response time which is counted, but does not terminate before the end of the nth 0-cycle. Since the point estimates and confidence intervals here are based on large sample theory (strong laws and central limit theorems), these errors are negligible for n large. In fact, the errors due to the two response times at t=0 and at

the end of the simulation run compensate for each other. Again, we have i.i.d. pairs of random variables $\{(\tau_k, N_k^{(2)}):k\geq 1\}$ and the ratio formula

$$E\{R^{(2)}\} = E\{\tau_1\}/E\{N_1^{(2)}\} \quad , \tag{3.2}$$

provided that $E\{R^{(2)}\}<\infty$. The point estimate of $E\{R^{(2)}\}$ is $\bar{\tau}/\bar{N}^{(2)}$, and we can use the standard regenerative method to obtain a confidence interval.

Estimation of $E\{R^{(1)}\}-E\{R^{(2)}\}$

Suppose now that we wish to estimate $r^{(1)}-r^{(2)}$, where $r^{(1)}=E\{R^{(1)}\}$ and $r^{(2)}=E\{R^{(2)}\}$. We can take as a point estimate the quantity $(\bar{\tau}/\bar{N}^{(1)})-(\bar{\tau}/\bar{N}^{(2)})$, but need a bivariate central limit theorem in order to produce a confidence interval. To this end, we let

$$Z_k^{(\nu)} = \tau_k - r^{(\nu)} N_k^{(\nu)}$$

and

$$\underset{\sim}{Z}_k = \begin{pmatrix} Z_k^{(1)} \\ \\ Z_k^{(2)} \end{pmatrix}$$

for $k\geq 1$. (We take all our vectors to be column vectors.) The random vectors $\{\underset{\sim}{Z}_k:k\geq 1\}$ are i.i.d. since each $\underset{\sim}{Z}_k$ is only a function of the kth 0-cycle. Furthermore, equations (3.1) and (3.2) imply that $E\{\underset{\sim}{Z}_k\}=\underset{\sim}{0}$. Denoting the transpose of $\underset{\sim}{Z}_k$ by $\underset{\sim}{Z}_k'$, let $\underset{\sim}{\Sigma}=E\{\underset{\sim}{Z}_k\underset{\sim}{Z}_k'\}=\{\sigma_{ij}\}$ be the covariance matrix of the $\underset{\sim}{Z}_k$'s. Assuming that the elements of $\underset{\sim}{\Sigma}$ are finite, we have the central limit theorem

$$n^{-1/2}\sum_{k=1}^{n}\underset{\sim}{Z}_k \Rightarrow N(\underset{\sim}{0},\underset{\sim}{\Sigma}) \quad , \tag{3.3}$$

where $N(\underset{\sim}{0},\underset{\sim}{\Sigma})$ is a multivariate normal random variable with zero mean vector and covariance matrix $\underset{\sim}{\Sigma}$. We can rewrite equation (3.3) in the form

$$n^{1/2}\begin{bmatrix}\overline{N}^{(1)}\{(\overline{\tau}/\overline{N}^{(1)})-r^{(1)}\}\\ \overline{N}^{(2)}\{(\overline{\tau}/\overline{N}^{(2)})-r^{(2)}\}\end{bmatrix} \Longrightarrow N(\underset{\sim}{0},\underset{\sim}{\Sigma}) \quad . \qquad (3.4)$$

Since $\overline{N}^{(\nu)}\to E\{N_1^{(\nu)}\}$ with probability one, we can replace $\overline{N}^{(1)}$ and $\overline{N}^{(2)}$ outside the braces by $E\{N_1^{(1)}\}$ and $E\{N_1^{(2)}\}$ in equation (3.4) and not change the result. This is an application of the continuous mapping theorem (c.m.t.); see Billingsley [1], Theorem 5.1. Next apply the c.m.t. to this altered form of equation (3.3) using the continuous mapping $h(x_1,x_2)=(x_1/E\{N_1^{(1)}\},\ x_2/E\{N_1^{(2)}\})$ to obtain

$$n^{1/2}\begin{bmatrix}(\overline{\tau}/\overline{N}^{(1)})-r^{(1)}\\ (\overline{\tau}/\overline{N}^{(2)})-r^{(2)}\end{bmatrix} \Longrightarrow N(\underset{\sim}{0},\underset{\sim}{a}\underset{\sim}{\Sigma}\underset{\sim}{a}') \qquad (3.5)$$

where $\underset{\sim}{a}=(1/E\{N_1^{(1)}\},\ 1/E\{N_1^{(2)}\})$. Note that from equation (3.5) we could construct a simultaneous confidence interval for $(r^{(1)},r^{(2)})$. Finally, a third application of the c.m.t. yields

$$(n^{1/2}/\sigma)\ [\{(\overline{\tau}/\overline{N}^{(1)})-(\overline{\tau}/N^{(2)})\}-(r^{(1)}-r^{(2)})] \Longrightarrow N(0,1) \qquad (3.6)$$

where

$$\sigma^2 = \sigma_{11}/E^2\{N_1^{(1)}\} + \sigma_{22}/E^2\{N_1^{(2)}\} - 2\sigma_{12}/(E\{N_1^{(1)}\}E\{N_1^{(2)}\}) \quad .$$

We can use the central limit theorem of equation (3.6) to construct a confidence interval for $r^{(1)}-r^{(2)}$, provided that an estimate for the constant σ is available. Using the classical method (cf., Iglehart [5]), we can estimate σ from the sequence of observations taken in the n 0-cycles of the process $\underset{\sim}{V}$. This estimate for σ appears in the Appendix.

A special case of the situation just discussed is when the two types of jobs are the same; in this case there is only one job type, but we elect to mark two jobs. Let $r^{(1)}=r^{(2)}=r$, $\hat{r}_n^{(\nu)}=\bar{\tau}/N^{(\nu)}$, and $\underset{\sim}{\hat{r}}_n=(\hat{r}_n^{(1)},\hat{r}_n^{(2)})$. Then we can use the method of multiple estimates of Heidelberger [4] applied to equation (3.5). For any vector $\underset{\sim}{\beta}=(\beta_1,\beta_2)$ with $\beta_1+\beta_2=1$, we have

$$(n^{1/2}/\sigma(\underset{\sim}{\beta}))\ (\underset{\sim}{\beta}'\hat{r}_n-r) \Longrightarrow N(0,1) \quad ,$$

where $\sigma^2(\underset{\sim}{\beta})=\underset{\sim}{\beta}'(\underset{\sim}{a}\underset{\sim}{\Sigma}\underset{\sim}{a}')\underset{\sim}{\beta}$. Next we select that value of $\underset{\sim}{\beta}$, call it $\underset{\sim}{\beta}^*$, which minimizes $\sigma^2(\underset{\sim}{\beta})$ subject to $\underset{\sim}{\beta}'\underset{\sim}{e}=1$, where $\underset{\sim}{e}=(1,1)$. It turns out that $\underset{\sim}{\beta}^*$ is given by

$$\underset{\sim}{\beta}^* = \{1/(\underset{\sim}{e}'\underset{\sim}{a}\underset{\sim}{\Sigma}\underset{\sim}{a}'\underset{\sim}{e})\}(\underset{\sim}{a}\underset{\sim}{\Sigma}\underset{\sim}{e}')^{-1}\underset{\sim}{e}$$

and

$$\sigma^2(\underset{\sim}{\beta}) = 1/\{\underset{\sim}{e}'(\underset{\sim}{a}\underset{\sim}{\Sigma}\underset{\sim}{a}')^{-1}\underset{\sim}{e}\} \quad . \tag{3.7}$$

Since $\underset{\sim}{\beta}=(1,0)$ is one possible value of $\underset{\sim}{\beta}$, using $\underset{\sim}{\beta}^*$ is guaranteed to yield a variance reduction over that obtained by marking just one job. Again, of course, we must estimate the variance $\sigma^2(\underset{\sim}{\beta}^*)$ given in equation (3.7) from the observations recorded.

Estimation of $P\{R^{(1)}\le x\}-P\{R^{(2)}\le x\}$

Finally, we consider the estimation of $P\{R^{(1)}\le x\}-P\{R^{(2)}\le x\}$ for a given value of x. This is the most difficult of the problems we treat. Since the value of x is fixed throughout the discussion, in general we suppress in our notation the dependence of x. Again we form 0-cycles based on the response times for the type 1 marked job. Here, however, when a 0-cycle ends, we do not know whether the response time for the type 2 marked job

in progress will be less than or equal to x. Thus, with respect to the response times for the type 2 marked job, the 0-cycles used previously do not create the i.i.d. blocks needed to establish a central limit theorem. Instead, we form new cycles by grouping together a geometrically distributed number of consecutive 0-cycles. Let $T_i=\tau_1+\ldots\tau_i$, $i\geq 1$, and $T_0=0$. We let s_i denote the time of the last entrance of $\underset{\sim}{V}$ into $S^{(2)}$ during the ith 0-cycle; s_i is the start time of the response time for the type 2 marked job underway at the conclusion of the ith 0-cycle. Set $A_i=\{T_i-s_i>x\}$ and $p_i=P\{A_i\}$. Since the 0-cycles are i.i.d., the events $\{A_i:i\geq 1\}$ are also, and $p_i=p$ for all $i\geq 1$. We assume that at the start of the simulation the value of the response time for the type 2 marked job underway is greater than (the fixed) x. This corresponds to the beginning of one of the new "super-cycles" we are constructing. If we let γ denote a geometric random variable with distribution given by

$$P\{\gamma=n\} = (1-p)^{n-1}p,\ n\geq 1 \quad ,$$

then the length of the first super-cycle is simply $t_1=\tau_1+\tau_2+\ldots+\tau_\gamma$. The number of response times for the type ν marked job started in this super-cycle is $n_1^{(\nu)}=N_1^{(\nu)}+\ldots+N_\gamma^{(\nu)}$. Successive super-cycles are defined in an analogous fashion. Define the sequence $\{Y_k^{(\nu)}:k\geq 1\}$ to be the number of response times terminating in the kth super-cycle which are less than or equal to x; for example

$$Y_1^{(\nu)} = \sum_{k=0}^{n_1^{(\nu)}-1} 1_{\{R_k^{(\nu)}\leq x\}} \quad .$$

Observe that by the definition of a super-cycle, the first response time of the type 2 marked job terminating within a super-cycle must be greater

than x. Thus the sequence of $Y_k^{(2)}$'s are i.i.d. Of course, the $Y_k^{(1)}$'s are i.i.d. also, as are the $n_k^{(1)}$'s and $n_k^{(2)}$'s. We can now form the bivariate central limit theorem analogous to equation (3.5), namely

$$n^{1/2}\begin{bmatrix}(\bar{Y}^{(1)}/\bar{n}^{(1)})-P\{R^{(1)}\le x\}\\ (\bar{Y}^{(2)}/\bar{n}^{(2)})-P\{R^{(2)}\le x\}\end{bmatrix} \Longrightarrow N(\underset{\sim}{0},\underset{\sim}{b}(x)\underset{\sim}{\Sigma}(x)\underset{\sim}{b}'(x)) ,$$

where $\underset{\sim}{b}(x)=(1/E\{n_1^{(1)}\},\ 1/E\{n_1^{(2)}\})$, and $\underset{\sim}{\Sigma}(x)=\{\sigma_{ij}(x)\}$ with

$$\sigma_{ij}(x) = E\{[Y_1^{(i)}-n_1^{(i)}P\{R^{(i)}\le x\}][Y_1^{(j)}-n_1^{(j)}P\{R^{(j)}\le x\}]\} .$$

Finally, by the same argument used in equation (3.6), we obtain

$$(n^{1/2}/\sigma(x))[(\bar{Y}^{(1)}/\bar{n}^{(1)}-\bar{Y}^{(2)}/\bar{n}^{(2)})-(P\{R^{(1)}\le x\}-P\{R^{(2)}\le x\})]$$

$$\Longrightarrow N(0,1) , \tag{3.8}$$

where

$$\sigma^2(x) = (\sigma_{11}(x)/E^2\{n_1^{(1)}\})+(\sigma_{22}(x)/E^2\{n_1^{(2)}\})-(2\sigma_{12}(x)/E\{n_1^{(1)}\}E\{n_1^{(2)}\}) .$$

We can estimate the quantity $\sigma(x)$ from the observations in the n super-cycles using the classical method; see the Appendix. Then we construct confidence intervals for $P\{R^{(1)}\le x\}-P\{R^{(2)}\le x\}$ from equation (3.8) in the usual way.

4. EXAMPLE AND NUMERICAL RESULTS

To illustrate the technique of the previous section for estimation of response times, we consider a simple closed network of queues having two types of jobs and two service centers; see Figure 1. There are N jobs in the network, N_1 jobs of type 1 and N_2 jobs of type 2. Upon completion of service in center 1, a type ν job joins the queue at center 1 (with

probability $\psi^{(\nu)}$) or (with probability $1-\psi^{(\nu)}$) joins the queue in center 2. Upon completion of service at center 2, jobs join the queue at center 1. At both service centers, type 1 jobs have non-preemptive priority over type 2 jobs. Jobs of the same type at either of the centers receive service in order of their arrival at the center. We assume that all service times are mutually independent; jobs of type ν at center i receive service which is exponentially distributed with parameter $\lambda_i^{(\nu)}$. The (limiting) response time $R^{(\nu)}$ for type ν jobs that we consider in this model is the time measured from when upon completion of service at center 2, a type ν job enters the queue at center 1, until the next such entrance by the job into the queue at center 1.

In this model, there are two job classes, class 1 jobs at center 1 and class 2 jobs at center 2. Each center sees both job types, but only one job class. The irreducible Markov routing matrices $\underline{P}^{(\nu)}$ are of the form

$$\underline{P}^{(\nu)} = \begin{bmatrix} \psi^{(\nu)} & 1-\psi^{(\nu)} \\ 1 & 0 \end{bmatrix} .$$

Since type 1 jobs have priority over type 2 jobs at both centers, the (type, class) pairs ordered by decreasing priority are $j_1(i)=(1,i)$ and $j_2(i)=(2,i)$, $i=1,2$. For this model, it is sufficient to take as the component $S_i(t)$ in the vector $Z(t)$ the type of job in service at center i at time t, rather than the (type, class) pair. Then we can define the vector $Z(t)$ as

$$Z(t) = (C_1^{(2)}(t),C_1^{(1)}(t),S_1(t),C_2^{(2)}(t),C_2^{(1)}(t),S_2(t)) ,$$

where, for $i=1,2$ and $\nu=1,2$,

$C_i^{(\nu)}(t)$ = number of type ν jobs in queue at center i at time t,

and

$S_i(t)$ = type of job in service at center i at time t
0 if center i is idle at time t .

Letting $N_\nu(t)$, $\nu=1,2$ denote the position from the top of the type ν marked job in the linear job stack, the state vector for this model is

$$X(t) = (Z(t),N_1(t),N_2(t)), \ t\geq 0 .$$

Letting L(t) denote the last state visited by the Markov chain $\underset{\sim}{X}=\{X(t):t\geq 0\}$ before jumping to X(t), the vector V(t) is

$$V(t) = (L(t),X(t)), \ t\geq 0 .$$

For N=2 jobs, the state space E of the process $\{X(t):t\geq 0\}$ has six states and is

$$E = \{(0,0,0,1,0,1,2,1), (0,0,0,0,1,2,1,2), (0,0,1,0,0,2,1,2)\}$$
$$\cup \{(0,0,2,0,0,1,2,1), (1,0,1,0,0,0,2,1), (0,1,2,0,0,0,1,2)\} .$$

The subsets $A_1^{(1)}$ and $A_2^{(1)}$ of E defining the start of response times for the type 1 marked job are

$$A_1^{(1)} = \{(0,0,0,1,0,1,2,1), (0,0,2,0,0,1,2,1)\}$$
$$A_2^{(1)} = \{(0,0,1,0,0,2,1,2), (0,1,2,0,0,0,1,2)\} .$$

Similarly, the subsets $A_1^{(2)}$ and $A_2^{(2)}$ of E defining the start of response times for the type 2 marked job are

$$A_1^{(2)} = \{(0,0,0,0,1,2,1,2),\ (0,0,1,0,0,2,1,2)\}$$
$$A_2^{(2)} = \{(0,0,2,0,0,1,2,1),\ (1,0,1,0,0,0,2,1)\}\ .$$

Since $R^{(1)}$ and $R^{(2)}$ are response times, $B_1^{(\nu)}=A_1^{(\nu)}$ and $B_2^{(\nu)}=A_2^{(\nu)}$, $\nu=1,2$; see Figure 2. It is easy to check that the state space F of the process $\{V(t):t\geq 0\}$ has nine states. The subsets $S^{(\nu)}$ of F defining the starts of response times for the type ν marked job are

$$S^{(1)} = \{(0,0,0,1,0,1,2,1,0,0,1,0,0,2,1,2)\}$$
$$\cup\ \{(0,0,2,0,0,1,2,1,0,1,2,0,0,0,1,2)\}$$

and

$$S^{(2)} = \{(0,0,0,0,1,2,1,2,0,0,2,0,0,1,2,1)\}$$
$$\cup\ \{(0,0,1,0,0,2,1,2,1,0,1,0,0,0,2,1)\}\ ,$$

respectively; see Figure 3. Here we use the enumeration of the six states of E given in Figure 2. Thus, e.g., (1,3) denotes the state $(0,0,0,1,0,1,2,1,0,0,1,0,0,2,1,2)\epsilon F$.

Results obtained by simulation of this model for $\psi^{(1)}=\psi^{(2)}=.75$, $\lambda_1^{(1)}=\lambda_1^{(2)}=\lambda_1=1$ and $\lambda_2^{(1)}=\lambda_2^{(2)}=\lambda_2=.5$, with N=2, appear in Tables 1-4. With these parameter values, there is one type 1 job and one type 2 job. The routing and service requirements of the two job types are the same; the two jobs differ only with respect to the non-preemptive priority given (at each center) to the type 1 job. The simulation used the congruential uniform random number generator described by Lewis, Goodman, and Miller

[11], with exponential service times obtained by logarithmic transformation of the uniform random numbers. Independent streams of exponential random numbers (obtained from different seeds) were used to generate individual exponential holding time sequences.

In the simulation results of Tables 1-4, the return state defining 0-cycles of the response time for the type 1 job is the state (0,0,2,0,0,1,2,1,0,1,2,0,0,0,1,2). This corresponds to a response time for the type 1 (marked) job starting when the type 2 (marked) job is in service at center 1. Table 1 summarizes results of the simulation and reports point estimates and 90 percent confidence intervals for the quantities $E\{R^{(1)}\}$, $E\{R^{(2)}\}$ and $E\{R^{(1)}\}-E\{R^{(2)}\}$ over a range of number of cycles of the type 1 marked job. Theoretical values for these quantities are shown in parentheses. Thus, for example, 100 cycles of the type 1 marked job were observed in the simulated time interval (0,903.00) and there were a total of 446 transitions in the continuous time Markov chain $\{L(t):t\geq 0\}$. A total of 130 response times for the type 1 (marked) job were observed along with 56 response times for the type 2 (marked) job. For the quantity $E\{R^{(1)}\}=7$, the point estimate 6.946 was obtained, and the 90 percent confidence interval had half-length 0.6334. Note that for $E\{R^{(1)}\}$ and $E\{R^{(2)}\}$, all of the confidence intervals surrounded the theoretical values. In the case of $E\{R^{(1)}\}-E\{R^{(2)}\}$, the confidence intervals based on equation (3.6) also surrounded the theoretical value. Table 2 gives results obtained for $P\{R^{(1)}\leq x\}$, with x=4, 8, 12, 16 and 20.

In Table 3 we give, for the several values of x, point and interval estimates for $P\{R^{(1)}\le x\}-P\{R^{(2)}\le x\}$, based on the use of super-cycles and equation (3.8). Thus, for x=4, 100 cycles based on response times for the type 1 job resulted in 37 super-cycles defined by response times for the type 2 job greater than x. Note that the number of cycles for the type 1 marked job has been fixed, and for each x the estimates for $P\{R^{(1)}\le x\}-P\{R^{(2)}\le x\}$ computed from the resulting random number of super-cycles.

Table 4 contains estimates of the quantities $P\{R^{(2)}\le x\}$ obtained from the standard regenerative method applied to these super-cycles. An overall observation from Tables 2 and 4 is that the lengths of confidence intervals obtained for $P\{R^{(1)}\le x\}$ and $P\{R^{(2)}\le x\}$ are roughly comparable.

5. CONCLUDING REMARKS

The discussion in Section 3 concentrated on problems associated with the estimation of characteristics of response times for the two types of jobs. The estimation of characteristics of two passage times, or one response time and one passage time, is in general easier. This is because there is the possibility of forming, from 0-cycles based on one type of job, super-cycles which terminate when no passage time of the other type of job is underway.

We have considered explicitly only the case of two job types. The estimation methods of Section 3 apply equally well to networks having more

than two job types. The state space which results from the augmentation of the vector X(t) (by components to track a marked job of each of the job types) is of course larger.

TABLE 1

Simulation Results for Response Times in Closed Networks of Queues
With Two Job Types. $N_1=1$, $N_2=1$, $\psi=.75$, $\lambda_1=1$, $\lambda_2=.5$.
Return State is (0,0,2,0,0,1,2,1,0,1,2,0,0,0,1,2).

	No. of Cycles For Type 1 Marked Job				
	100	200	400	800	1000
Total simulated time	903.00	1959.09	4134.11	8211.57	10172.83
No. of transitions/ cycle (M.C.)	4.46	4.85	5.18	5.12	5.15
No. of type 1 response times/cycle	1.30	1.40	1.48	1.46	1.47
No. of type 2 response times/cycle	0.56	0.65	0.75	0.74	0.74
$E\{R^{(1)}\}$ (=7)	6.946 ±0.6334	6.997 ±0.3931	6.995 ±0.2703	7.018 ±0.2033	6.920 ±0.1800
$E\{R^{(2)}\}$ (=14)	16.125 ±2.8713	15.187 ±1.8017	13.780 ±0.9851	13.965 ±0.7361	13.729 ±0.6305
$E\{R^{(1)}\}-E\{R^{(2)}\}$ (=-7)	-9.179 ±2.7421	-8.190 ±1.7316	-6.785 ±0.9145	-6.947 ±0.6809	-6.808 ±0.5848

TABLE 2

Percentiles of Type 1 Response Times in Closed Networks of Queues With Two Job Types. $N_1=1$, $N_2=1$, $\psi=.75$, $\lambda_1=1$, $\lambda_2=.5$. Return State is (0,0,2,0,0,1,2,1,0,1,2,0,0,0,1,2).

	No. of Cycles For Type 1 Marked Job				
	100	200	400	800	1000
$P\{R^{(1)}\leq 4\}$	0.2384 ±0.0622	0.2536 ±0.0417	0.2555 ±0.0301	0.2641 ±0.0217	0.2639 ±0.0192
$P\{R^{(1)}\leq 8\}$	0.6692 ±0.0683	0.6714 ±0.0444	0.6717 ±0.0308	0.6709 ±0.0221	0.6802 ±0.0201
$P\{R^{(1)}\leq 12\}$	0.8923 ±0.0422	0.8786 ±0.0293	0.8832 ±0.0205	0.8769 ±0.0159	0.8830 ±0.0140
$P\{R^{(1)}\leq 16\}$	0.9461 ±0.0311	0.9536 ±0.0198	0.9594 ±0.0135	0.9547 ±0.0105	0.9605 ±0.0088
$P\{R^{(1)}\leq 20\}$	0.9923 ±0.0127	0.9892 ±0.0100	0.9915 ±0.0061	0.9880 ±0.0052	0.9898 ±0.0043

TABLE 3

Difference of Percentiles of Response Times in Closed Networks of Queues With Two Job Types. $N_1=1$, $N_2=1$, $\psi=.75$, $\lambda_1=1$, $\lambda_2=.5$. Return State is (0,0,2,0,0,1,2,1,0,1,2,0,0,0,1,2).

	No. of Cycles For Type 1 Marked Job				
	100	200	400	800	1000
$P\{R^{(1)}\le 4\}-P\{R^{(2)}\le 4\}$	0.1254 ±0.0784	0.1295 ±0.0627	0.1342 ±0.0417	0.1332 ±0.0275	0.1384 ±0.0192
No. of super-cycles	37	79	181	347	438
$P\{R^{(1)}\le 8\}-P\{R^{(2)}\le 8\}$	0.3988 ±0.1227	0.3409 ±0.0817	0.3111 ±0.0525	0.3036 ±0.0373	0.3125 ±0.0201
No. of super-cycles	29	61	134	254	319
$P\{R^{(1)}\le 12\}-P\{R^{(2)}\le 12\}$	0.3915 ±0.1424	0.3543 ±0.1112	0.3331 ±0.0677	0.3259 ±0.0496	0.3280 ±0.0140
No. of super-cycles	22	46	93	180	224
$P\{R^{(1)}\le 16\}-P\{R^{(2)}\le 16\}$	0.2850 ±0.1288	0.2970 ±0.1092	0.2693 ±0.0606	0.2693 ±0.0489	0.2636 ±0.0088
No. of super-cycles	15	32	65	129	153
$P\{R^{(1)}\le 20\}-P\{R^{(2)}\le 20\}$	0.2422 ±0.1081	0.2470 ±0.0825	0.2119 ±0.0530	0.2078 ±0.0415	0.2009 ±0.0043
No. of super-cycles	11	24	43	84	104

TABLE 4

Percentiles of Type 2 Response Times in Closed Networks of Queues With Two Job Types. $N_1=1$, $N_2=1$, $\psi=.75$, $\lambda_1=1$, $\lambda_2=.5$. Return State is $(0,0,\bar{2},0,0,1,2,1,0,1,2,\bar{0},0,0,\bar{1},2)$.

	No. of Cycles For Type 1 Marked Job				
	100	200	400	800	1000
$P\{R^{(2)}\le 4\}$	0.1071 ±0.0852	0.1240 ±0.0583	0.1200 ±0.0312	0.1310 ±0.0209	0.1255 ±0.0183
No. of super-cycles	37	79	181	347	438
$P\{R^{(2)}\le 8\}$	0.2679 ±0.0963	0.3281 ±0.0751	0.3600 ±0.0475	0.3673 ±0.0323	0.3681 ±0.0280
No. of super-cycles	29	61	134	254	319
$P\{R^{(2)}\le 12\}$	0.5000 ±0.1184	0.5234 ±0.0799	0.5500 ±0.0474	0.5510 ±0.0336	0.5548 ±0.0286
No. of super-cycles	22	46	93	180	224
$P\{R^{(2)}\le 16\}$	0.6607 ±0.1155	0.6563 ±0.0687	0.6900 ±0.0419	0.6854 ±0.0316	0.6969 ±0.0268
No. of super-cycles	15	32	65	129	153
$P\{R^{(2)}\le 20\}$	0.7500 ±0.0986	0.7422 ±0.0582	0.7793 ±0.0405	0.7802 ±0.0284	0.7889 ±0.0243
No. of super-cycles	11	24	43	84	104

APPENDIX

We first consider estimation of the variance constant σ^2 appearing in equation (3.6) which leads to a confidence interval for $E\{R^{(1)}\}-E\{R^{(2)}\}$. Based on n cycles, for i=1,2, compute $\hat{\sigma}_{ii}$ as an estimate of

$$\sigma_{ii} = E\{(Z_k^{(1)})^2\} = \mathrm{var}\{\tau_k^{(1)}\}-2r^{(1)}\mathrm{cov}\{\tau_k^{(1)},N_k^{(1)}\}+(r^{(1)})^2\mathrm{var}\{N_k^{(1)}\}$$

according to

$$\hat{\sigma}_{ii} = s_{11} - 2\hat{r}^{(1)}s_{12}^{(i)} + (\hat{r}^{(1)})^2 s_{22}^{(i)} \quad ,$$

where

$$s_{11} = \frac{1}{n-1}\sum_{j=1}^{n}(\tau_j-\bar{\tau})^2 \quad ,$$

$$s_{12}^{(i)} = \frac{1}{n-1}\sum_{j=1}^{n}(\tau_j-\bar{\tau})(N_j^{(i)}-\bar{N}^{(i)}) \quad ,$$

and

$$s_{22}^{(i)} = \frac{1}{n-1}\sum_{j=1}^{n}(N_j^{(i)}-\bar{N}^{(i)})^2 \quad ,$$

with

$$\bar{\tau} = \frac{1}{n}\sum_{j=1}^{n}\tau_j \ , \ \bar{N}^{(i)} = \frac{1}{n}\sum_{j=1}^{n}N_j^{(i)} \text{ and } \hat{r}^{(i)} = \bar{\tau}/\bar{N}^{(i)} \quad .$$

Finally, compute $\hat{\sigma}_{12}$ as an estimate of

$$\sigma_{12} = \mathrm{var}\{\tau_k\}-r^{(1)}\mathrm{cov}\{\tau_k,N_k^{(1)}\}-r^{(2)}\mathrm{cov}\{\tau_k,N_k^{(2)}\}$$
$$+ r^{(1)}r^{(2)}\mathrm{cov}\{N_k^{(1)},N_k^{(2)}\}$$

according to

$$\hat{\sigma}_{12} = s_{11} - \hat{r}^{(1)}s_{12}^{(1)} - \hat{r}^{(2)}s_{12}^{(2)} + \hat{r}^{(1)}\hat{r}^{(2)}s_{22} ,$$

where s_{11}, $s_{12}^{(1)}$ and $s_{12}^{(2)}$ are as before, and

$$s_{22} = \frac{1}{n-1}\sum_{j=1}^{n}(N_j^{(1)}-\bar{N}^{(1)})(N_j^{(2)}-\bar{N}^{(2)}) .$$

Then estimate σ^2 according to

$$\hat{\sigma}^2 = \frac{\hat{\sigma}_{11}}{(\bar{N}^{(1)})^2} + \frac{\hat{\sigma}_{22}}{(\bar{N}^{(2)})^2} - \frac{2\hat{\sigma}_{12}}{\bar{N}^{(1)}\bar{N}^{(2)}} .$$

In an analogous manner, we estimate the variance constant $\sigma^2(x)$ appearing in equation (3.8) which leads to a confidence interval for $P\{R^{(1)}\le x\}-P\{R^{(2)}\le x\}$. Based on n super-cycles, for $i=1,2$, compute $\hat{\sigma}_{ii}(x)$ as an estimate of

$$\hat{\sigma}_{ii}(x) = \mathrm{var}\{Y_k^{(i)}\}-2P\{R^{(i)}\le x\}\mathrm{cov}\{Y_k^{(i)},n_k^{(i)}\} + (P\{R^{(i)}\le x\})^2\mathrm{var}\{n_k^{(i)}\}$$

according to

$$\hat{\sigma}_{ii}(x) = s_{11}^{(i)}(x) - 2\left(\frac{\bar{Y}^{(i)}}{\bar{n}^{(i)}}\right)s_{12}^{(i)}(x) + \left(\frac{\bar{Y}^{(i)}}{\bar{n}^{(i)}}\right)^2 s_{22}^{(i)}(x)$$

where

$$s_{11}^{(i)}(x) = \frac{1}{n-1}\sum_{j=1}^{n}(Y_j^{(i)}-\bar{Y}^{(i)})^2 ,$$

$$s_{12}^{(i)}(x) = \frac{1}{n-1}\sum_{j=1}^{n}(Y_j^{(i)}-\bar{Y}^{(i)})(n_j^{(i)}-\bar{n}^{(i)}) ,$$

and

$$s_{22}^{(i)}(x) = \frac{1}{n-1}\sum_{j=1}^{n}(n_j^{(i)}-\bar{n}^{(i)})^2 ,$$

with

$$\overline{Y}^{(i)} = \frac{1}{n}\sum_{j=1}^{n} Y_j^{(i)} \text{ and } \overline{n}^{(i)} = \frac{1}{n}\sum_{j=1}^{n} n_j^{(i)} \quad .$$

Finally, compute $\hat{\sigma}_{12}(x)$ as an estimate of

$$\sigma_{12}(x) = \mathrm{cov}\{Y_k^{(1)},Y_k^{(2)}\} - P\{R^{(1)} \le x\}\mathrm{cov}\{Y_k^{(2)},n_k^{(1)}\}$$
$$- P\{R^{(2)} \le x\}\mathrm{cov}\{Y_k^{(1)},n_k^{(2)}\}$$
$$+ P\{R^{(1)} \le x\}P\{R^{(2)} \le x\}\mathrm{cov}\{n_k^{(1)},n_k^{(2)}\}$$

according to

$$\hat{\sigma}_{12}(x) = s_{11}^{(1)}(x) - \left(\frac{\overline{Y}^{(1)}}{\overline{n}^{(1)}}\right) s_{12}^{(1)}(x) - \left(\frac{\overline{Y}^{(2)}}{\overline{n}^{(2)}}\right) s_{21}^{(2)}(x)$$
$$+ \left(\frac{\overline{Y}^{(1)}\overline{Y}^{(2)}}{\overline{n}^{(1)}\overline{n}^{(2)}}\right) s_{22}(x) \quad ,$$

where $s_{11}^{(1)}(x)$, $s_{22}^{(1)}(x)$ and $s_{12}^{(2)}(x)$ are as before, and

$$s_{22}(x) = \frac{1}{n-1}\sum_{j=1}^{n}(n_j^{(1)}-\overline{n}^{(1)})(n_j^{(2)}-\overline{n}^{(2)}) \quad .$$

Then estimate $\sigma^2(x)$ according to

$$\hat{\sigma}^2(x) = \frac{\hat{\sigma}_{11}(x)}{(\overline{n}^{(1)})^2} + \frac{\hat{\sigma}_{22}(x)}{(\overline{n}^{(2)})^2} - \frac{2\hat{\sigma}_{22}(x)}{\overline{n}^{(1)}\overline{n}^{(2)}} \quad .$$

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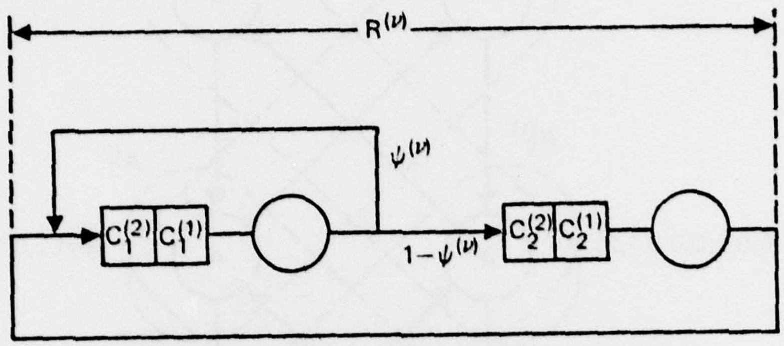

(i) Services at centers 1 and 2 are not interruptable

(ii) Routing for type ν jobs determined by binary valued variable $\psi^{(\nu)}$

(iii) Type 1 jobs have non-preemptive priority over type 2 jobs

Figure 1. Closed network of queues with two job types.

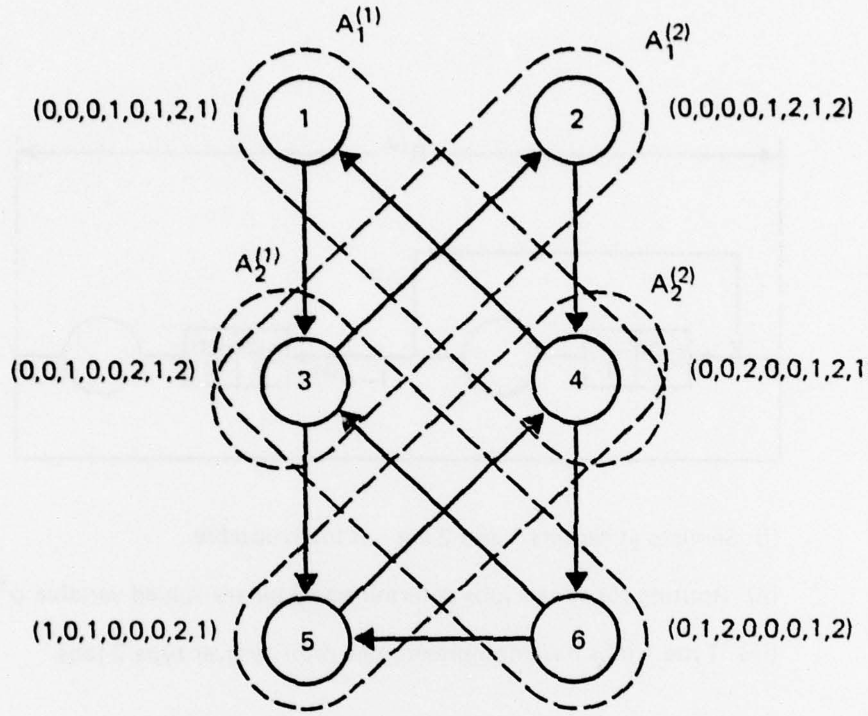

Figure 2. State transitions in Markov chain X and subsets of E for response times $R^{(1)}$ and $R^{(2)}$.

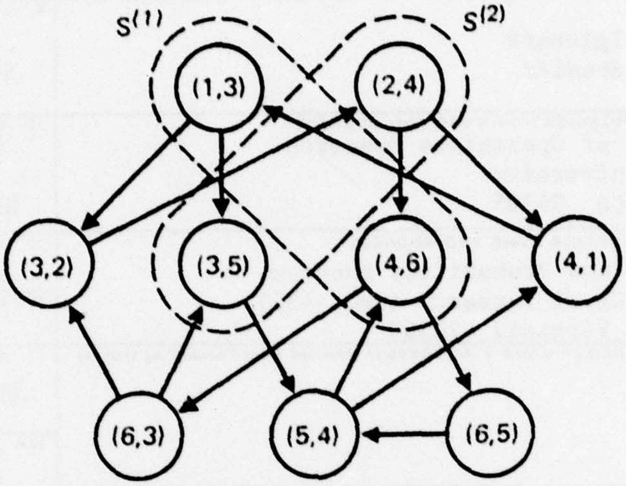

Figure 3. State transitions in Markov chain V and subsets of F for response times $R^{(1)}$ and $R^{(2)}$.

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20. ABSTRACT.

REGENERATIVE SIMULATION OF RESPONSE TIMES IN NETWORKS OF QUEUES, II: MULTIPLE JOB TYPES - Technical Report No. 48

We have previously discussed the simulation of networks of queues for general characteristics of passage times of a single job type, using the regenerative method for simulation and the idea of tracking a distinguished job through the network. We consider here, from a somewhat different point of view, passage time simulation in closed networks of queues having multiple job types. Our results provide a means of obtaining, from a single replication, point and interval estimates for passage times of the several job types. They also yield a statistically more efficient estimation procedure for passage times of a single job type.